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# TILINGS OF THE SQUARE WITH SIMILAR RIGHT TRIANGLES

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We prove that if $\triangle$ is a right triangle with an acute angle $\alpha$ where tan $\alpha$ is a totally positive algebraic number then the square can be decomposed into finitely many similar copies of $\triangle$. This result completes the classification of all triangles which "tile" the square.

## Introduction

We will use the notation of [1], so we will say that a triangle $\triangle$ tiles the polygon $P$ if $P$ can be decomposed into finitely many non-overlapping triangles similar to $\triangle$. L. Pósa asked the following question: Does the triangle with angles $30°, 60°, 90°$ tile the square? M. Laczkovich answered this question in [1] (The answer is no). Moreover he proved the following two theorems:

**Theorem 1.** (Laczkovich) *If a right triangle $\triangle$ with an acute angle $\alpha$ tiles the square then tan $\alpha$ is a totally positive algebraic number.*

**Theorem 2.** (Laczkovich) *If a triangle $\triangle$ tiles the square and $\triangle$ is not a right triangle then the angles of $\triangle$ are $(\pi/8, \pi/4, 5\pi/8)$ or $(\pi/4, \pi/3, 5\pi/12)$ or $(\pi/12, \pi/4, 2\pi/3)$. (And each of the three types tiles the square.)*

We will prove the reverse of Theorem 1 and so we get the full classification of triangles which tile the square. For this purpose we use a theorem from [2]:

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**Theorem 3.** (Laczkovich–Szekeres) *A rectangle of eccentricity $v$ tiles the square if and only if $v$ is an algebraic number and each conjugate of $v$ has a positive real part.*

## Notation, lemmas

Let $R(x)$ denote the rectangle with sides $x$ and 1, furthermore let $T(u)$ denote the right triangle with perpendicular sides $u$ and 1. We define the following set: $k(u) := \{x \mid T(u)$ tiles the rectangle $R(x)\}$.

**Lemma A.** *We have the following rules for $k(u)$:*

A.1) $u \in k(u)$;

A.2) *If* $x, y \in k(u)$ *then* $x + y \in k(u)$;

A.3) *If* $x \in k(u)$ *then* $1/x \in k(u)$;

A.4) *If $r$ is an arbitrary positive rational number and* $x \in k(u)$ *then* $rx \in k(u)$;

A.5) *If* $x \in k(u)$ *and $n$ is a positive integer then* $n(\frac{1+u^2}{x+u}) + u \in k(u)$.

**Proof.** The statements A.1, A.2, A.3, A.4 are easy to see (cf. [2]). For proving A.5 we define the trapezoid $N(a,u)$ with two parallel sides of length $a$ and $a+u$ so that a third side of length 1 is perpendicular to the two paralell sides. (See Figure 1.)

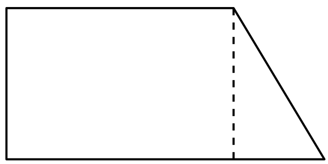


*Fig. 1*

Let $s(u) := \{a \mid T(u)$ tiles $N(a,u)\}$. We prove that $x, y \in s(u)$ implies $\frac{1+u^2}{x+u} + y \in s(u)$. As shown in Figure 2. we can see that the trapezoid $N\left(\frac{1+u^2}{x+u} + y, u\right)$ can be decomposed into a similar copy of $T(u)$ , $N(x,u)$ and $N(y,u)$ so the previous statement is clear.

Our following step to prove A.5 is to show that $x \in s(u)$ implies $n\frac{1+u^2}{x+u} \in s(u)$ for an arbitrary positive integer $n$. That goes by induction on $n$. In the case $n = 1$ the statement follows immediately from the previous rule applying it for the case of $x$,0. (Note that $0 \in s(u)$ because $N(0,u)$ equals

$T(u)$) Assume that the statement is true for $n$ namely that $n\frac{1+u^2}{x+u} \in s(u)$. We apply our rule for $x$ and $y = n\frac{1+u^2}{x+u}$ and so we get the statement for $n+1$.

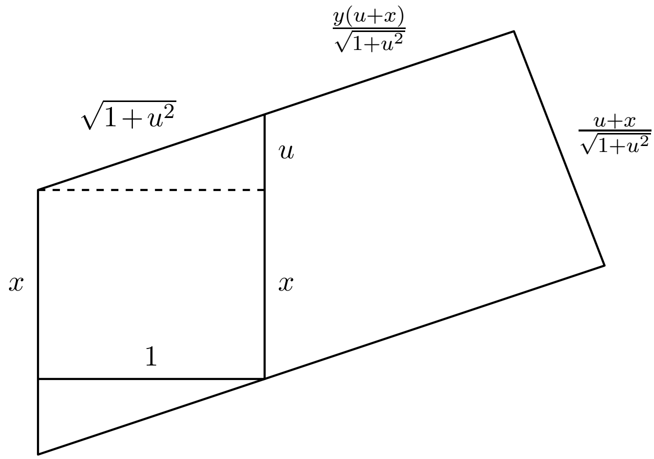


*Fig. 2*

**The proof of A.5.** If $x \in k(u)$ then $x \in s(u)$ because the trapezoid $N(x,u)$ can be decomposed into $T(u)$ and $R(x)$. From $x \in s(u)$ we get that $n\frac{1+u^2}{x+u} \in s(u)$ for an arbitary positive integer $n$. Finally we get that $n\frac{1+u^2}{x+u} + u \in k(u)$ through completing the trapezoid $N(n\frac{1+u^2}{x+u}, u)$ to the rectangle $R(n\frac{1+u^2}{x+u} + u)$ by the triangle $T(u)$. ∎

Let $\mathcal{P} \subset \mathbb{Q}(t)$ denote the unique smallest subset which satisfies the conditions A.1, A.2, A.3, A.4, A.5 with $t$ instead of $u$ and $\mathcal{P}$ instead of $k(u)$. ($\mathbb{Q}(t)$ denotes the set of rational functions over $\mathbb{Q}$ in one indeterminant $t$) The following statement is a corollary of Lemma A:

**Corollary A.** *If* $u \in \mathbb{R}^+$ *and* $f \in \mathcal{P}$ *then* $f(u) \in k(u)$.

Let $\overline{\mathcal{P}}$ denote the closure of $\mathcal{P}$ in $\mathbb{R}(t)$ corresponding to the pointwise convergence in $\mathbb{C}$. That means that $f \in \overline{\mathcal{P}}$ if and only if $f \in \mathbb{R}(t)$ and there exists a sequence $f_n \in \mathcal{P}$ so that $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in \mathbb{C}$ where $f$ is defined.

**Lemma B.** *If* $p(t)$ *is a polynomial with nonnegative real coefficients then* $t \cdot p(1+t^2) \in \overline{\mathcal{P}}$.

**Proof.** We define the $g_n^k(t)\in\mathbb{Q}(t)$ rational functions recursively on $k$:

$$g_n^1(t) := \frac{1+t^2}{\frac{1}{t}+\frac{t}{n}} + \frac{t}{n}$$
$$g_n^k(t) := \frac{1+t^2}{\frac{1}{g_n^{k-1}(t)}+\frac{t}{n}} + \frac{t}{n}$$

First we prove by induction that $g_n^k(t)\in\mathcal{P}$. For this purpose it is enough to prove that $f\in\mathcal{P}$ implies:

$$\frac{1+t^2}{\frac{1}{f}+\frac{t}{n}} + \frac{t}{n} \in \mathcal{P}$$

It is easy because:

$$f\in\mathcal{P} \Rightarrow \frac{1}{f}\in\mathcal{P} \Rightarrow \frac{n}{f}\in\mathcal{P} \Rightarrow$$
$$n^2\frac{1+t^2}{\frac{n}{f}+t} + t\in\mathcal{P} \Rightarrow n\frac{1+t^2}{\frac{n}{f}+t} + \frac{t}{n}\in\mathcal{P} \Rightarrow \frac{1+t^2}{\frac{1}{f}+\frac{t}{n}} + \frac{t}{n}\in\mathcal{P}$$

(We have used here the rules A.3, A.4, A.5, A.4)

We obtain, by induction on k, that $\lim_{n\to\infty} g_n^k(z)=z(1+z^2)^k$ for all $z\in\mathbb{C}$ and $k\geq 1$. This shows us that $t(1+t^2)^k\in\overline{\mathcal{P}}$. It is clear that $\overline{\mathcal{P}}$ is closed under multiplication with positive real numbers and addition (see A.4, A.2) and this completes our proof. ∎

Let $\Re(x)$ denote the real-part and $\Im(x)$ the imaginary-part of a complex number $x$.

**Lemma C.** *Let $S$ be a finite set of distinct complex numbers such that $S\cap\overline{S}=\emptyset$. ($\overline{S}$ is the set of complex conjugates of the elements of $S$) Let $f:S\to\mathbb{C}$ be an arbitrary function. Then there exists a polynomial $p(t)$ with nonnegative real coefficients such that $p(x)=f(x)$ for all $x\in S$.*

**Proof.** First we will find a polynomial $g(t)\in\mathbb{R}[t]$ such that $g(x)=f(x)$ for all $x\in S$. It is clear that for all $x\in S$ there exists a linear polynomial $g_x(t)\in\mathbb{R}(t)$ so that $g_x(x)=f(x)/\prod_{\substack{y\in S\\ y\neq x}}(x-y)(x-\overline{y})$ because $x$ and 1 generates $\mathbb{C}$ as a vector space over $\mathbb{R}$. The polynomial $g(t)$ can be chosen as follows:

$$g(t) := \sum_{x\in S} g_x(t)\prod_{\substack{y\in S\\ y\neq x}}(t-y)(t-\overline{y}).$$

For each $x \in S$ there exists a natural number $n_x$ such that $\Re(x^{n_x}) < 0$, because $x$ is not a positive number. We define the following polynomial:

$$h(t) := \prod_{x \in S} (|x^{n_x}|^2 + t^{n_x}|2\Re(x^{n_x})| + t^{2n_x}).$$

The polynomial $h(t)$ has non-negative real coefficients and $h(x)=0$ for all $x \in S$. Let $p_{a,k}(t) = g(t) + a \cdot h(t)(1+t+t^2+\ldots+t^k)$. Now $p_{a,k}(t) \in \mathbb{R}[t]$ and $p_{a,k}(x) = f(x)$ for all $x \in S$. If $a \in \mathbb{R}^+$ and $k \in \mathbb{N}$ are big enough, then the coefficients of $p_{a,k}(t)$ are non-negative because the first $k$ coefficients of $a \cdot h(t)(1+t+t^2+\ldots+t^k)$ are bigger then $a \cdot h(0)$ and $h(0)>0$. ∎

**Lemma D.** *If $u$ is a positive algebraic number and each of its real conjugates is positive then there exists a polynomial $p(t)$ with non-negative real coefficients such that $v \cdot p(1+v^2)>0$ for each conjugates $v$ of $u$.*

**Proof.** Let $H$ be the set of all non-real conjugates of $u$. Note that if $x,y \in H$ then $1+x^2=1+y^2$ implies $x=y$ because if $x=-y$ then $u=x^\varphi=-y^\varphi$ for some Galois automorphism $\varphi$ but $y^\varphi=-u<0$ contradicts our assumption. In particular, $x \in H$ implies $\Im(1+x^2) \neq 0$ because $\overline{x} \in H$ and so $1+x^2 \neq 1+\overline{x}^2 = \overline{1+x^2}$.

Let $H_1$ be a subset of $H$ such that $H_1 \cup \overline{H}_1 = H$ , $H_1 \cap \overline{H}_1 = \emptyset$. Using Lemma C we get a polynomial $p(t)$ with non-negative real coefficients such that $p(1+x^2)=1/x$ for all $x \in H_1$ and so $p(1+x^2)=1/x$ for all $x \in \overline{H}_1$ because $1/\overline{x} = \overline{p(1+x^2)} = p(1+\overline{x}^2)$. Summarizing the previous statements we arrive at $x \cdot p(1+x^2)=1$ for all $x \in H$. If $v$ is a real conjugate of $u$ then $v>0$ and so $v \cdot p(1+v^2)>0$ because $p(t)$ is not the zero polynomial. ∎

**Theorem.** *If $u$ is a positive algebraic number such that each of its real conjugates is positive then the triangle $T(u)$ tiles the square.*

**Proof.** According to Lemma D there exists a polynomial $p(t)$ with non-negative real coefficients such that $v \cdot p(1+v^2)>0$ for all conjugates $v$ of $u$. Using Lemma B we get that $t \cdot p(1+t^2) \in \overline{\mathcal{P}}$ and so there exists a sequence $f_n(t) \in \mathcal{P}$ such that $\lim_{n\to\infty} f_n(x) = x \cdot p(1+x^2)$ for all $x \in \mathbb{C}$. The number of conjugates of $u$ is finite so we get that there exists $m \in \mathbb{N}$ such that $\Re(f_m(v)) > 0$ for all conjugates $v$ of $u$. Since $f_m(t) \in \mathbb{Q}(t)$, the conjugates of $f_m(u)$ are the numbers $f_m(v)$ where $v$ runs over the conjugates of $u$. It means that $f_m(u)$ is an algebraic number such that each of its conjugates has a positive real-part. From Theorem 3 (Laczkovich–Szekeres [2]) it follows that $R(f_m(u))$ tiles the square and from Corollary A it follows that $T(u)$ tiles $R(f_m(u))$. ∎

## References

[1] M. Laczkovich: Tilings of polygons with similar triangles, *Combinatorica*, **10** (1990), 281–306.
[2] M. Laczkovich and G. Szekeres: Tiling of the square with similar rectangles, *Discrete Comput. Geom.,* **13** (1995), 569–572.

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